◎ Matrix Operations

☆ Example

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + z\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} x + 2y + 4z \\ -2x + 5y + z \\ -4x + y + 2z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} \underline{c_1} & \underline{c_2} & \underline{c_3} \end{bmatrix}$$

$$\underline{c_2} = x_2\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + y_2\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + z_2\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} = a\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} + b\begin{bmatrix} -2 & 3 & 1 \end{bmatrix} + c\begin{bmatrix} -4 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} a-2b-4c & 2a+3b+c & 4a+b+2c \end{bmatrix}$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \underline{r_1} \\ \underline{r_2} \\ \underline{r_3} \end{bmatrix}$$

$$\underline{r_3} = a_3\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} + b_3\begin{bmatrix} -2 & 3 & 1 \end{bmatrix} + c_3\begin{bmatrix} -4 & 1 & 2 \end{bmatrix}$$

$A(BC) = (AB)C$ (Associative law holds)

$AB \neq BA$ (Commutative law does not hold)

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$C(A+B) = CA + CB$

(Distributive laws hold)

$(A+B)C = AC + BC$

◎ Elimination Using Matrices

☆ Example

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$\times -2 \quad \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= E_{21}} \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right]$$

E_{21} (substract a multiple 2 of row 1 from row 2)

elementary matrix
elimination matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right]$$

$$\times 1 \quad \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{= E_{31}}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= E_{21}}\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right]$$

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{= P_{23}}\begin{bmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{bmatrix}$$

P_{23} (exchange rows 2 and 3)

permutation matrix
(row exchange matrix)

$$\begin{aligned} x + 2y + 2z &= 1 \\ 4x + 8y + 9z &= 3 \\ 3y + 2z &= 1 \end{aligned}$$

$$[\,A\ \underline{b}\,] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [\,A\ \underline{b}\,] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{= P_{23}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= E_{21}} [\,A\ \underline{b}\,] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

◎ Inverse Matrices

$\underset{\text{identity matrix}}{I} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n\times n}$ $\quad AI = IA = A$ for any $n\times n$ matrix A

Def An $n\times n$ matrix A is invertible if there exists a matrix B such that $BA = I$ and $AB = I$ (B is called an inverse of A)

Claim Suppose A is invertible. Then its inverse is unique.

Proof Suppose A has two inverse B and C. Then $BA = I$ and $AC = I$.
We have $B = BI = B(AC) = (BA)C = IC = C$ ∎

Remark The inverse of A is denoted as A^{-1}.

Remark The proof actually shows that if $BA = AC = I$ then $B = C = A^{-1}$.
"left inverse" = "right inverse" = "inverse"

Claim The inverse of A^{-1} is A itself.

Proof $AA^{-1} = I$ (since A^{-1} is the inverse of A) and $A^{-1}A = I$ (since A^{-1} is the inverse of A)
$\therefore A$ is the inverse of A^{-1}. ∎

Claim If A is invertible, the one and only one solution to $A\underline{x} = \underline{b}$ is $\underline{x} = A^{-1}\underline{b}$

Proof $A\underline{x} = \underline{b} \Leftrightarrow A^{-1}A\underline{x} = A^{-1}\underline{b} \Leftrightarrow I\underline{x} = A^{-1}b \Leftrightarrow \underline{x} = A^{-1}\underline{b}$ ∎

Claim Suppose there is a nonzero solution $\underline{x}$ to $A\underline{x} = \underline{0}$.
Then A cannot have an inverse.

Proof If A is invertible, $A\underline{x} = \underline{0} \Rightarrow (A^{-1}A)\underline{x} = A^{-1}\underline{0} \Rightarrow \underline{x} = \underline{0}$. ∎

Claim A diagonal matrix has an inverse provided no diagonal entries are zero.

Proof If $A = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 1/d_1 & & 0 \\ & \ddots & \\ 0 & & 1/d_n \end{bmatrix}$. ∎

Claim If A and B are invertible, then so is AB.
$(AB)^{-1} = B^{-1}A^{-1}$

Proof
$$\begin{aligned}(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}IB \\ &= B^{-1}B = I\end{aligned}$$

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} = I\end{aligned}$$ ∎

Claim $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Example
$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

elementary matrix
(elimination matrix)

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example
$$P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

permutation matrix
(row exchange matrix)

$$P_{21}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P_{21}$$